Compressed Sensing and Its Applications to Magnetic Resonance Imaging (MRI)

Frankie Camacho
Sponsoring Scientist: Professor Jan S. Hesthaven, PhD
Division of Applied Mathematics, Brown University, Providence, RI

BROWN

Introduction

- Conventional sampling theory demands that one sample measurements at the Nyquist rate (twice the maximum frequency).
- Compressed sensing allows for the recovery of compressible signals and images in a particular basis by using a significantly fewer number of measurements versus what is required by classical methods.
- Compressive sensing relies on a signal's sparsity and a large incoherence between the sensing and measurement bases.
- We can transform the problem into a convex optimization problem and harness a host of numerical methods.
- We use our algorithm on several toy and real world problems, i.e. 1 and 2 dimensional reconstructions (with and without noise).
- We explore techniques to enhance the machinery's robustness.

Mathematical Setup

Basis-signal relationship is

$$f = \Psi x, \qquad \Psi \in \mathbb{C}^{N\times N}, x \in \mathbb{C}^{N\times 1}, \tag{1}$$

$$y = \Phi f, \qquad \Phi \in \mathbb{C}^{m\times N}, \tag{2}$$

where f is the original signal we want to consruct, Ψ is the sparsifying basis with a k-sparse coefficient vector x, Φ is the measurement basis, and y is the measurement vector. $y = \Phi f$ is an underdetermined linear system, so there is no unique solution, *unless we assume that x is sparse.*

ℓ_1 Minimization

Since we want the sparsest solution x that satisfies our linear constraints, the problem is equivalent to solving

$$\min \|x\|_0 \qquad \text{subject to } \Phi\Psi x = y, \tag{3}$$

where $\|x\|_0$ is the number of non-zero entries in x. Unfortunately, ℓ_0 minimization is an *NP*-hard combinatorial problem, which means numerical methods won't work for even moderately large N. However, by replacing the ℓ_0 norm with the ℓ_1 norm, i.e.

$$\min \|x\|_1 \qquad \text{subject to } \Phi\Psi x = y, \tag{4}$$

compressive sensing theory guarantees exact recovery of the sparsest solution with probability $\rho = 1 - O(N^{-m})$, where N is the length of the signal or image, and m is the number of measurements.

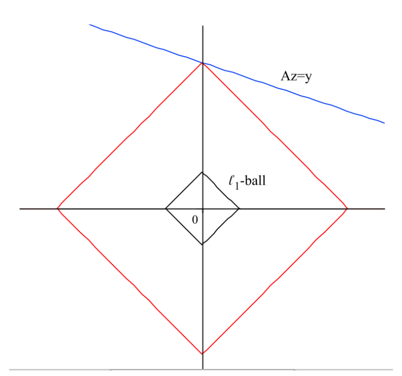

Figure: "The ℓ_1 minimizer within the affine space of solutions of the linear system $\Phi\Psi x = y$ coincides with a sparsest solution." [Fornasier, M, and Rauhut, H. Compressive Sensing. 18 April 2010.]

How Compressive Sensing Works

Restricted Isometry Property: Let $A = \Phi\Psi$. Then the RIP constant is the smallest number δ_k such that

$$(1-\delta_k)\|x\|_2^2 \leqslant \|Ax\|_2^2 \leqslant (1+\delta_k)\|x\|_2^2, \tag{5}$$

for all k-sparse vectors x. The smaller the δ_k, the closer A is to becoming an isometric operator. If $\delta_k \in (0,1)$ then A satisfies a RIP condition and we are guaranteed a unique solution via the ℓ_1 minimization. That is,

Theorem: If A satisfies the RIP condition with RIP constant $\delta_{3k} < \frac{1}{3}$ and x^* is the ℓ_1 solution, then

$$\|x - x^*\|_2 = 0. \tag{6}$$

Measurements: In order to satisfy the RIP condition, one can take *uniformly random* measurements to guarantee that this property holds with overwhelmingly high probability.

1-D Reconstruction

Below is an example of a 1-D reconstruction.

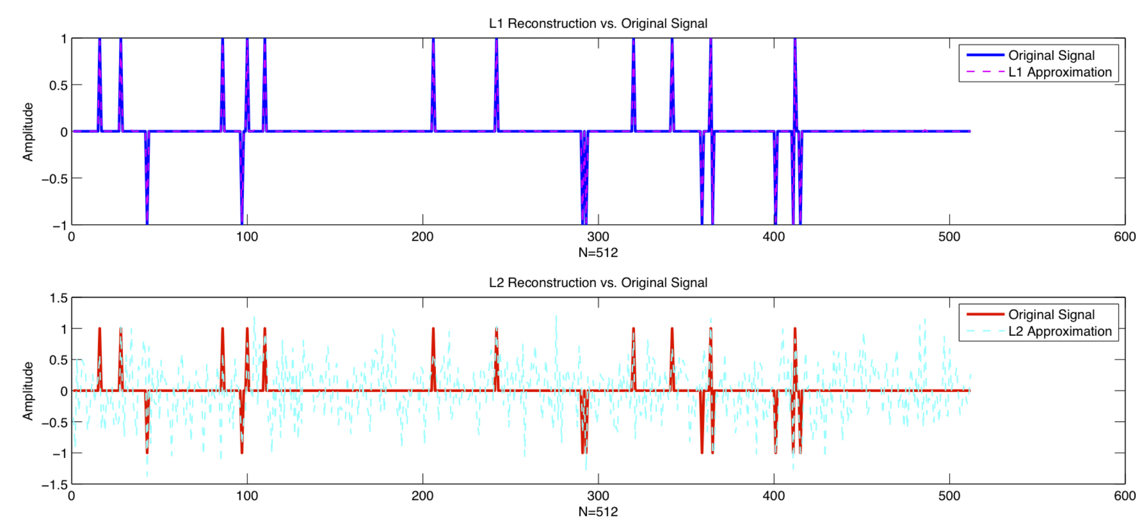

Figure: Top: Reconstruction of a 20-sparse signal x of length 512 using ℓ_1 minimization and only 20% data from a random Gaussian measurement matrix ($G_{i,j} = \frac{1}{m}X$, where $X \sim \mathcal{N}(0,1)$). Bottom: Here we compare with the ℓ_2 solution. Note that the ℓ_2 solution is far from correct.

Applications to MRI's

Reconstruction of the brain from MRI data is a type of inverse problem that could greatly benefit from the application of compressive sensing. In many cases, the brain MRI signal is sparse in the total variation domain, so we can under-sample its Fourier information ($\Phi_{kj} = \frac{1}{\sqrt{N}} e^{\frac{2\pi i k j}{N}}$) and use compressive sensing to retrieve the signal.

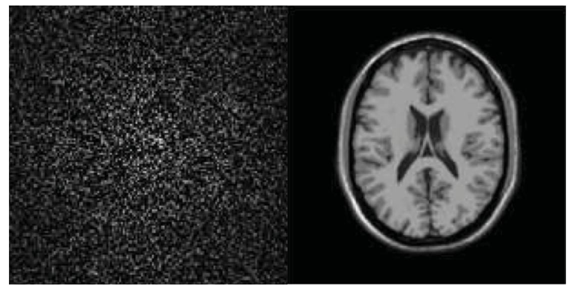

Figure: One example of how compressive sensing might be used to recreate MRI's. (left) Fourier transform information regarding the signal. (right) Reconstruction using compressive sensing.

2-D Shepp-Logan Reconstruction

Below are some sample results of a 256×256 Shepp-Logan Phantom image reconstruction using $m = \frac{n}{4}$ (25%) measurements, i.e. $O(\log^4 n)$ where $n = 256^2$. The image is sparse in the total variation domain.

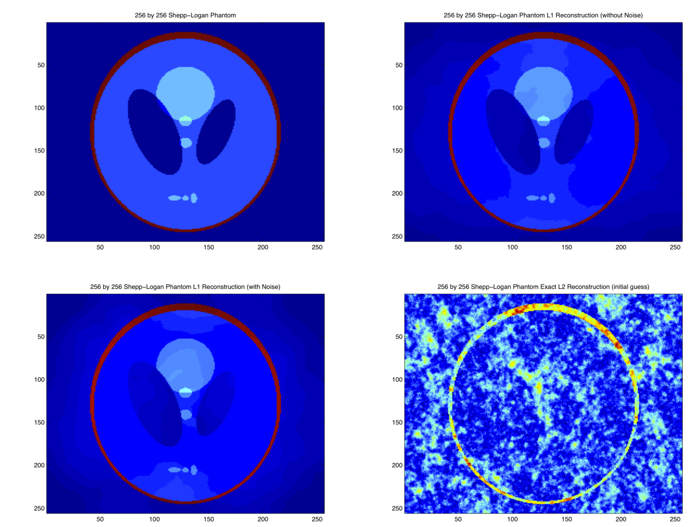

Figure: Top: (left) The Shepp-Logan Phantom image, widely used in the field of tomography, of size 256×256. (right) The ℓ_1 reconstruction without noise. Bottom: (left) The ℓ_1 reconstruction with real Gaussian noise added. (right) The initial guess that is fed to our nonlinear conjugate gradient method with Newton kick-step algorithm is the ℓ_2 exact reconstruction given by (4).

Error Analysis

Below is a graphical analysis of the behavior of the relative error between the original $N \times N$ Shepp-Logan Phantom image with $N = 32, 64$, and its corresponding ℓ_1 reconstruction with noise.

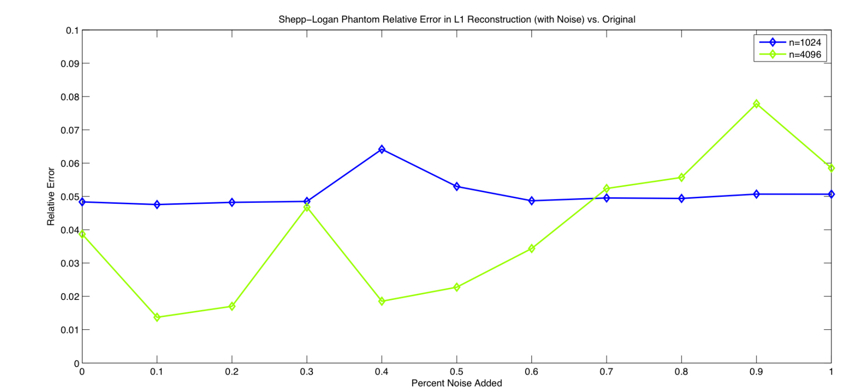

Figure: Error analysis for the relative error $\left(\frac{\|X_{\text{true}} - X_{\text{reconstructed}}\|_2}{\|X_{\text{true}}\|_2}\right)$ of the Shepp-Logan Phantom image. The results, while not always proportional, fall within the bounds given by compressive sensing theory (i.e. the relative error should be proportional to the noise, $\|y - Ax\|_2$).

Future Projects

Going forward, there are different optimization methods that we would like to implement in our research. For example, there are numerous existing ℓ_1 solvers (ex. SPGS) that are emerging in literature that would potentially give more accurate and faster reconstructions. Additionally, we would like to implement code in lower level languages like C++ to reduce computational costs and improve efficiency, and utilize parallelization and GPU computing techniques in order to gain speedup. Finally, because real data is not provided on a cartesian grid (which we need in order to utilize the FFT), we would like to test these results with Fourier coefficients obtained via re-gridding techniques, which map non-cartesian data to cartesian data.

camacho.frankie@gmail.com